How to Model a Frequency Modulated (FM) Signal – a quick insight

- Some time back when I first attempted to model a PLL, I needed to use the FM signal from the output of the VCO in my numerical setup. In my naivety I first tried to use the straight formula of phase function of frequency [Phase(time)=2*pi*freq.*time].

-This had a very positive effect since the horrendously looking waveform made me really think hard about what I did wrong and how I could accomplish my goal correctly. It took me few hours to figure it out alone.

- Had I read this in a book I would have had a hard time truly understanding this "little detail" and its impact in most of my future work.

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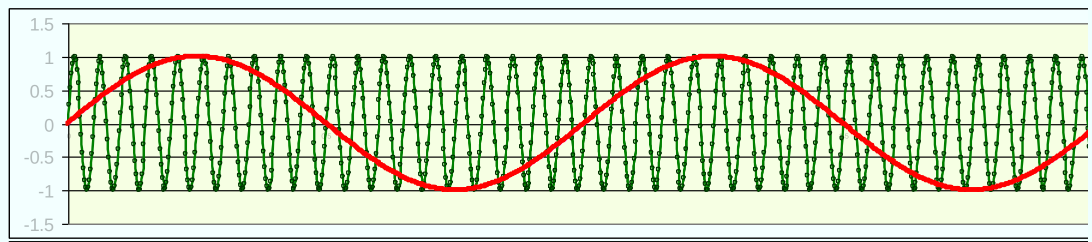
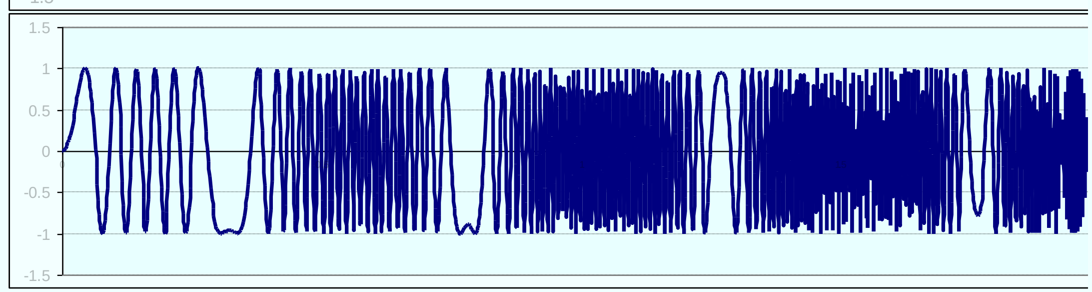
My original attempt (the wrong way):

- We know from school that a time varying sinusoidal signal (with initial phase zero) and frequency which is dependent of time can be written as: $u(t) = \sin[2 \cdot \pi \cdot f(t) \cdot t]$

  - Let's assume that the frequency function has the following expression in which f_0 is a high frequency (carrier) modulated by a lower frequency f_m: $f(t) = f_0 \cdot \sin(2 \cdot \pi \cdot f_m \cdot t)$

The expression of the final FM signal: $u(t) = \sin[2 \cdot \pi \cdot f_0 \cdot \sin(2 \cdot \pi \cdot f_m \cdot t) \cdot t]$



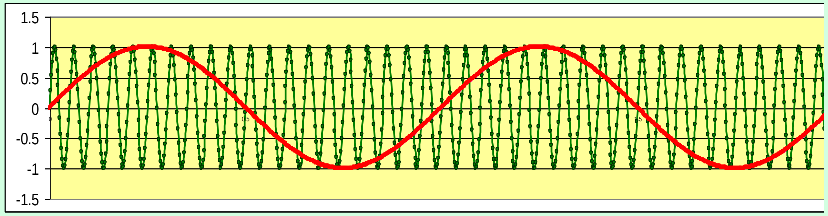


- If we correctly model the above equation, assuming f_0= 20Hz and f_m = 1Hz, we expect to see a signal with amplitude of 1, changing frequency periodically every second.
- On the top chart (yellow background) we can see the carrier and the modulating wave (red).
- The blue waveform is the modulated wave and it doesn't look as being periodic with a period of 1 sec. Also the second half of the modulated wave looks extremely messy with frequencies much higher than the carrier frequency. In this case the frequency of the modulated wave is supposed to oscillate between zero and 20 Hz depending of the value of the modulating wave. This wrong model is contained in the first worksheet, named “FM_Wrong_Way”.

What is wrong with the original model.

- We modeled three waves, **the first two** were constant frequency sinusoids and those were correctly charted.

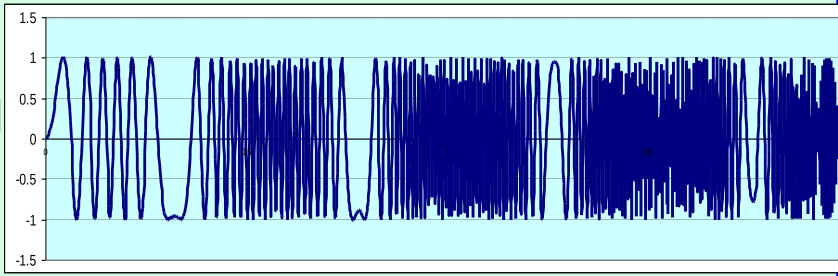
$$\begin{cases} u_0(t) = \sin\left(2\cdot\pi\cdot f_0\cdot t\right) & \text{for t = [0,h,2h,3h,….] - green curve} \\ u_m(t) = \sin\left(2\cdot\pi\cdot f_m\cdot t\right) & \text{for t = [0,h,2h,3h,….] - red curve} \end{cases}$$



- The third one was a sinusoidal wave with a variable frequency (the frequency variation seems to be the problem)

$$u(t) = \sin\Big[\overbrace{2\cdot\pi\cdot f_0\cdot \underbrace{\sin\left(2\cdot\pi\cdot f_m\cdot t\right)}_{\textit{Frequency(t) – time varying}}\cdot t}^{\textit{Phase}}\Big]$$

for t = [0,h,2h,3h,….] blue curve



It seems like at the beginning of the time interval the FM wave looks about right and the more the time advances, the worse the curve looks.

Explanation by analogy with linear movement:

- We know that frequency is the rate of oscillation, which means how many periods a periodic function covers in one second. In instantaneous terms the frequency is a scaled version of the time derivative of phase by the following formula:

$$f(t) = \frac{1}{2\cdot\pi}\cdot\frac{d\varphi(t)}{dt}$$

- As an analogy we can associate the phase in an oscillation to the distance in the linear movement. We can also associate the frequency in an oscillation to the velocity in linear movement. In linear movement, **unless the velocity is constant** (even if $t_{initial}=0$) **we cannot** calculate the final coordinate with the formula:

$$x_{final} = x_{initial} + v(t_{final}) \cdot t_{final}$$

it is an equality only over a small time interval while the speed is approximately constant.

So in general we have this: $x_{final} \neq x_{initial} + v(t_{final}) \cdot t_{final}$

- In a non-uniform linear movement (with known velocity function) we can approximate the coordinate at a certain time by splitting that time in small intervals *dt=h* over which the velocity can be approximated as constant and use the following finite difference expression to calculate the speed recursively:

$$x_n = x_{n-1} + v(n) \cdot h$$

or non-recursively if we wish so: $x_n = x_{initial} + h \cdot \sum_{i=0}^{n} v(i)$

How do we correct the original model?

- Applying the linear analogy to the oscillatory movement we can say that in an oscillation with variable frequency, we cannot model the final phase with the formula:

$$\varphi_{final} = \varphi_{initial} + 2 \cdot \pi \cdot f(t_{final}) \cdot t_{final}$$

- We can apply that formula only for uniform oscillation (constant frequency) or in the case of a variable frequency oscillation (FM) we can use the formula as an approximation over small time intervals:

$$\varphi_n = \varphi_{n-1} + 2 \cdot \pi \cdot f(n) \cdot h$$

Step-by-step example of the formula application:

- For time $t_0 = 0$ we enter the initial phase (I assumed it was zero): $\varphi_0 = 0$
- For time $t_1 = t_0 + h = h$ calculate the phase using the recurrent formula from the previous page: $\varphi_1 = \varphi_0 + 2 \cdot \pi \cdot f(h) \cdot h$
- For time $t_2 = t_1 + h = 2h$ calculate the phase using the recurrent formula from the previous page: $\varphi_2 = \varphi_1 + 2 \cdot \pi \cdot f(2h) \cdot h$
- For time $t_3 = t_2 + h = 3h$ calculate the phase using the recurrent formula from the previous page: $\varphi_3 = \varphi_2 + 2 \cdot \pi \cdot f(3h) \cdot h$
- For time $t_4 = t_3 + h = 4h$ calculate the phase using the recurrent formula from the previous page: $\varphi_4 = \varphi_3 + 2 \cdot \pi \cdot f(4h) \cdot h$

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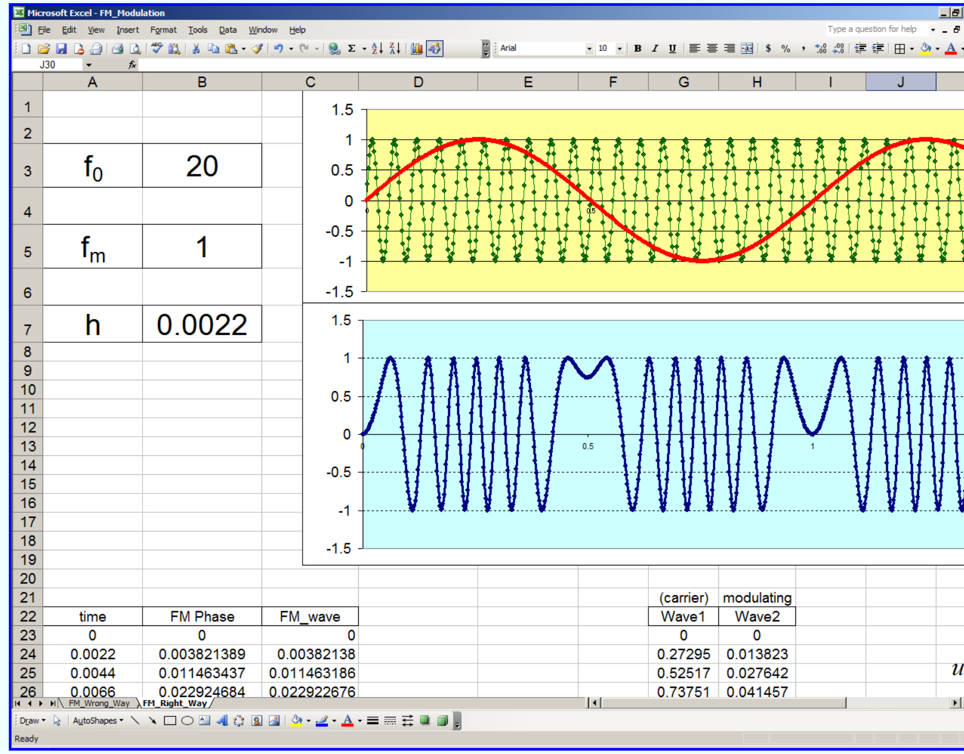
- For time $t_n = t_{n-1} + h = nh$ calculate the phase using the recurrent formula from the previous page: $\varphi_n = \varphi_{n-1} + 2 \cdot \pi \cdot f(nh) \cdot h$

The correct spreadsheet implementation of the FM signal model:

- The “FM_Modulation_Tutorial” workbook has two worksheets, the “FM_Wrong_Way” and “FM_Right_Way”. While they are quite similar, understanding of the first is left as an exercise to the reader. How do we implement “FM_Right_Way”?
- Cell B3 contains parameter f_0, cell B5 contains parameter f_m, cell B7 contains parameter h (time step).
- Range A22:C22 and range G22:H22 contain labels (table heads for the calculation area).

- Column G and H contain the carrier signal and the modulating signal respectively
- G23: “=SIN(2*PI()*B$3*A23)”
- H23: “=SIN(2*PI()*B$5*A23)”
- Copy range G23:H23 down to row 1023



- Column A contains an increasing time series (the time coordinate for all three signals)
- A23: “=0
- A24: “=A23+B$7”
- Copy cell A24 down to row 1023

- Column B contains the FM modulated phase
- B23: “=0
- B24: “=B23+2*PI()*B$3*SIN(2*PI()*B$5*A24)*B$7”
- Copy cell B24 down to row 1023

- Column C contains the FM signal
- C23: “=sin(B23)”
- Copy cell C23 down to row 1023

This presentation was an introduction to numerical modeling of FM signals. In Excel we generally need three columns to model such a signal. The modeling is not difficult but it has to be done in the proper fashion. In the PLL model we can save a column (the time column) since the modulation function takes its data from the VCO column.